Near-Wall k-ϵ Turbulence Modeling

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The flowfields from a turbulent channel simulation are used to compute the budgets for the turbulence kinetic energy (k) and its dissipation rate ϵ. Data from boundary-layer simulations are used to analyze the dependence of the eddy-viscosity damping function on the Reynolds number and the distance from the wall. The computed budgets are used to test existing near-wall turbulence models of the k-ϵ type. We find that the turbulent transport models should be modified in the vicinity of the wall. We also find that existing models for the different terms in the ϵ-budget are adequate in the region away from the wall but need modification near the wall. Our results indicate that in the context of k-ϵ modeling ad hoc eddy-viscosity damping functions play an important role in predicting the kinetic energy profile.

I. Introduction

OF the models used to predict turbulent flows, a popular model is the two-equation k-ϵ model. This model gained popularity with the advent of large computers and has been used extensively for engineering flows, even though it fails to predict correctly a number of flows.[1] The most commonly used model was developed for high-Reynolds-number flows and is used in wall-bounded flows in conjunction with wall functions to patch the core region of the flow to the wall region (for a review see Rodi[2]).

The usefulness of a k-ϵ turbulence model that would be valid all the way to the wall has been recognized early in the use of the model. Widely used models that use wall corrections to the standard k-ϵ model are the models of Jones and Launder[3] (hereafter J&L) and Chien[4] (hereafter CH). Several other models with wall corrections have been developed and tested for boundary-layer flows, each with proposals for damping functions and modification to the high-Reynolds-number k-ϵ model. Bernard[5] evaluated four models by computing the channel flow with these models and found that all models underpredict the peak in the turbulent kinetic energy characteristic of near-wall flows. He attributed the failure of the models to poor modeling of the pressure-diffusion term. His conclusion is based on comparisons with extrapolations from experimental data in the $y^+ < 10$ region. We shall see that the pressure diffusion term in the turbulent kinetic-energy budget is small compared to the other terms at all y^+. In a systematic study, Patel et al.[6] evaluated eight models for boundary-layer flow predictions. They found that the three models that are based on the k-ϵ model give similar results; the model of Lam and Bremhorst[7] with elaborate damping functions was not superior to the CH model or a slightly modified J&L model. They also state that further refinement to the existing models is needed for any of the models to be used with confidence. The suspect weakness of the model is believed to be the closure models used in the ϵ-equation. The current trend in modeling

the ϵ-equation is to introduce ad-hoc damping functions in the models of the different terms. An exact assessment of the models has not been possible because it is difficult to measure accurately the dissipation rate of the turbulent kinetic energy near the wall.

With the advent of both advanced numerical methods and large-scale computers, full simulation of turbulent flows at low Reynolds numbers (Re) are now possible. Of interest to the near-wall k-ϵ modeling are the recent simulations of Moser and Moin[8] of a curved channel flow, Spalart[9,10] of a flow over a flat plate, and Kim, Moin, and Moser[11] (hereafter KMM) of a channel flow. In the calculation of KMM, the turbulent flowfield for a channel flow at $Re = 3300$ (based on half the channel width and centerline velocity) has been computed and the results compared with experimental data. The physical realism of these computed turbulent flowfields has been validated by comparing with measured turbulence statistics as well as turbulence structural information obtained from laboratory experiments.

The data base of KMM is used to compute the budget of the turbulence kinetic energy k and the budget of the dissipation rate of k (ϵ). These budgets are used to test closures for these equations. The data of Spalart are used to analyze the dependence of the eddy-viscosity damping function on y^+ and the Reynolds number. Finally, results using an eddy-viscosity damping function with no wall corrections in the ϵ-equation show that the k-profile can be adequately predicted. However, to predict the ϵ-profile, wall corrections in the ϵ-equation are needed. The results presented in the paper are based on low to moderate Reynolds number data, and extrapolations to high Reynolds numbers or adverse pressure gradients should be done with care.

II. Averaged Equations

A. Mean Momentum Equation

The averaged Navier-Stokes equations for incompressible flows nondimensionalized with the wall variables $u_\tau (= \sqrt{\tau_w/\rho}$, the friction velocity) and ν (the kinematic viscosity) are written as follows:

$$U_{i,t} + (U_i U_j)_{,j} + (\overline{u_i' u_j'})_{,j} = -P_{,i}/\rho + U_{i,jj} \tag{1a}$$

$$U_{i,i} = 0 \tag{1b}$$

where U_i and u_i' reresent mean and fluctuating velocities, respectively, and P is the mean pressure. In most simple phenomenological models, a Bouissinesq approximation is

used to close Eq. (1). In this case, the terms $\overline{u_i'u_j'}$ are approximated as

$$-\overline{u_i'u_j'} = 2\nu_T S_{ij} - (2/3)k\delta_{ij} \qquad (2)$$

where $k = \overline{u_i'u_i'}/2$ is the turbulent kinetic energy, $S_{ij} = (U_{i,j} + U_{j,i})/2$ is the mean strain-rate tensor, ν_T is an eddy viscosity, and δ_{ij} is the Kronecker delta function.

The k-ϵ modelers approximate the eddy viscosity ν_T as follows (see, for example, Patel et al.[6]):

$$\nu_T = C_\mu f_\mu (k^2/\tilde{\epsilon}) \qquad (3)$$

where C_μ is a model constant, f_μ is a damping function, and $\tilde{\epsilon}$ is a modified dissipation rate of k. Note that $\tilde{\epsilon} = u_{i,j}'u_{i,j}' + D$ where D is a function that has the same dimension as $\epsilon = u_{i,j}'u_{i,j}'$; it is chosen so that $\tilde{\epsilon}$ vanishes at the wall and $\tilde{\epsilon} = \epsilon$ at high Reynolds numbers. Different near-wall k-ϵ models use different expressions for D. Note that in this approach, the problem was reduced from deriving equations describing the evolution of $\overline{u_i'u_j'}$ to deriving equations for k and ϵ.

B. k-Equation

For an incompressible flow, the exact equation governing the transport of k is

$$k_{,t} + (U_j k)_{,j} = P_k + T_k + \Pi_k + D_k - \epsilon \qquad (4)$$

where the different terms on the right-hand side are given as (rate of...),

Production:

$$P_k = -\overline{u_i'u_j'}S_{ij}$$

Turbulence transport:

$$T_k = -\tfrac{1}{2}\overline{(u_i'u_i'u_j')}_{,j}$$

Pressure diffusion:

$$\Pi_k = -(1/\rho)\overline{(u_i'p')}_{,i}$$

Viscous diffusion:

$$D_k = k_{,jj}$$

Dissipation:

$$\epsilon = \overline{u_{i,j}'u_{i,j}'}$$

Figure 1 shows the various terms computed from the channel data of KMM as a function of the wall variable y^+. The salient feature of this plot is that, away from the wall, the production rate is almost balanced by the dissipation rate. Close to the wall, the production rate and dissipation rate are still the dominant terms, but the turbulent transport rate and viscous diffusion rate are no longer negligible. Only the pressure diffusion rate remains negligible compared to the other terms. The turbulent diffusion rate has a positive peak at $y^+ = 6$ and a negative peak at $y^+ = 15$. The viscous diffusion rate and the dissipation rate are related by the identity

$$\epsilon' \equiv -\overline{u_i'u_{i,jj}'} = \epsilon - k_{,jj} \qquad (5)$$

At the wall, the left-hand side of Eq. (5) vanishes and the diffusion rate exactly balances the dissipation rate. The expansion of the left-hand side of Eq. (5) in Taylor series will yield $\epsilon' = O(y^+)$ in the vicinity of the wall, and, according to the data, it has a positive slope at the wall. Away from the wall, the second derivative of k is small compared to (or of the opposite sign as) ϵ, implying that ϵ' is positive throughout the channel. In fact, for homogeneous flows, $\epsilon' = \epsilon$.

Comparison of the budget data in Fig. 1 with the estimates using Laufer's data (see Townsend[12]) shows that both the turbulence transport and pressure-diffusion terms are overes-
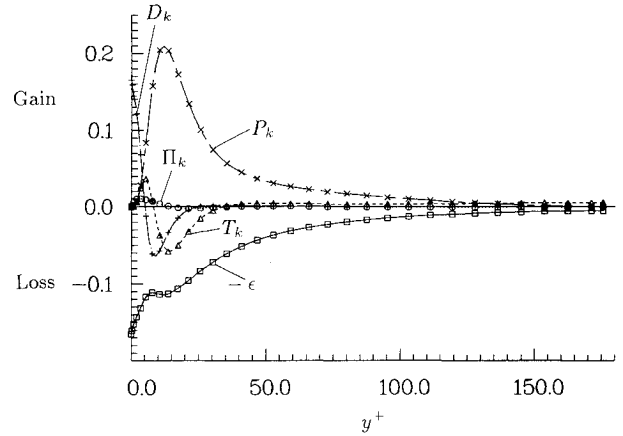

Fig. 1 Terms in the budget of the turbulence kinetic energy k in wall coordinate: P_k = production; T_k = turbulent transport; D_k = viscous diffusion; ϵ = dissipation rate; Π_k = velocity pressure gradient.

timated. The viscous diffusion at the wall is underestimated in Laufer's data, which yield a lower dissipation rate at the wall. Our data are consistent with the budget data of Moser and Moin[8] for a flow in a curved channel and of Spalart[9,10] for flows over a flat plate; close to the wall, all simulation data show that the pressure-diffusion term remains small compared to the other terms in the budget.

C. ϵ-Equation

The equation describing the evolution of ϵ is

$$\epsilon_{,t} + (U_j\epsilon)_{,j} = P_\epsilon^1 + P_\epsilon^2 + P_\epsilon^3 + P_\epsilon^4 + T_\epsilon + \Pi_\epsilon + D_\epsilon - \Upsilon \qquad (6)$$

We can identify the different terms on the right-hand side as (rate of ...),

Mixed production:

$$P_\epsilon^1 = -2\overline{u_{i,j}'u_{k,j}'}S_{ik}$$

Production by mean velocity gradient:

$$P_\epsilon^2 = -2\overline{u_{i,k}'u_{i,m}'}S_{km}$$

Gradient production:

$$P_\epsilon^3 = -2\overline{u_k'u_{i,m}'}U_{i,km}$$

Turbulent production:

$$P_\epsilon^4 = -2\overline{u_{i,k}'u_{i,m}'u_{k,m}'}$$

Turbulent transport:

$$T_\epsilon = -\overline{(u_k'u_{i,m}'u_{i,m}')}_{,k}$$

Pressure transport:

$$\Pi_\epsilon = -(2/\rho)\overline{(p_{,m}'u_{k,m}')}_{,k}$$

Viscous diffusion:

$$D_\epsilon = \epsilon_{,kk}$$

Dissipation:

$$\Upsilon = 2\overline{u_{i,km}'u_{i,km}'}$$

Tennekes and Lumley[13] analyzed the vorticity fluctuation budget, which is related to the above budget for homogeneous flows. They inferred from an order-of-magnitude analysis that in the high-Reynolds-number regime the turbulent-production rate P_ϵ^4 and dissipation rate Υ dominate the balance equation. However, the difference of these terms yields a term of the

order of the other terms. The various terms in the balance equation for ϵ are shown in Fig. 2. The present results indicate that P_ϵ^4 and Υ are the largest terms in the core region of the channel in agreement with the analysis of Tennekes and Lumley. Near the wall, these terms are still significant but are not larger than the other terms. Close to the wall $(y^+ < 8)$, the production rate P_ϵ^1 becomes of the same order as P_ϵ^4. In the range $6 < y^+ < 15$, the production rate by mean velocity gradient P_ϵ^2 is of the same order as P_ϵ^4.

III. Turbulence Models

In the previous section, we introduced the eddy-viscosity model to represent the Reynolds stresses and, in turn, modeled the eddy viscosity in terms of a damping function, the turbulence kinetic energy k, and the dissipation rate of k (ϵ). In turbulence modeling using k and ϵ, all the terms in the balance equations for k and ϵ that involve correlations other than k and ϵ have to be modeled in terms of U_i, k, and ϵ. In this section, we use the simulation data to show that the eddy-viscosity damping function should be a function of both the Reynolds number and y^+. We also use the simulation data to test closure models for the k and ϵ equations.

A. Damping the Eddy Viscosity

In flows where the relevant gradients are in one direction only (e.g., a fully developed channel or homogeneous shear), the Boussinesq approximation and the definition of the eddy viscosity will yield the following:

$$-\frac{\overline{u_1'u_2'}}{U_{,2}} = \nu_T = C_\mu f_\mu \frac{k^2}{\tilde{\epsilon}} \tag{7}$$

It can be shown that $\overline{u_1'u_2'} = \mathcal{O}(y^{+3})$, and $k = \mathcal{O}(y^{+2})$ as $y^+ \mapsto 0$. The choice of the near-wall behavior of the damping function f_μ will depend on the near-wall behavior of $\tilde{\epsilon}$. Using $D = -k_{,jj}$ will yield $\tilde{\epsilon} = \epsilon' = \mathcal{O}(y^+)$ in the vicinity of the wall.

The equation for the production rate of the turbulence kinetic energy $(P_k = -u_1'u_2' U_{,2})$ can be combined with the definition of the eddy viscosity and the Boussinesq approximation to yield

$$C_\mu f_\mu = \left(\frac{\overline{u_1'u_2'}}{k}\right)^2 \frac{\epsilon'}{P_k} \tag{8}$$

The damping function f_μ should be constructed by examining the behavior of the individual terms $-u_1'u_2'/k$ and P_k/ϵ' in wall-bounded flows. Figure 3 shows the distribution of $-u_1'u_2'/k$ as function of y^+ from the boundary layer data of Spalart[10] for three Reynolds numbers. The data show that $-u_1'u_2'/k$ is a function of the Reynolds number, and, for large y^+, the term seems to asymptote to a value close to 0.3,

$$-\overline{u_1'u_2'}/k = f_{uv}(y^+, Re) \tag{9}$$

The variation of P_k/ϵ' as a function of y^+ also was computed from the data of Spalart[9] and KMM (see Fig. 4). In the vicinity of the wall $(y^+ < 40)$, the data show a reasonable collapse and therefore exhibit dependence only on y^+,

$$P_k/\epsilon' = f_{p/\epsilon}(y^+) \tag{10}$$

The function $f_{p/\epsilon}$ peaks at around $y^+ = 19$ and $f_{p/\epsilon} = \mathcal{O}(y^{+2})$ in the vicinity of the wall.

B. k-Balance

In the k-equation, the viscous diffusion term need not be modeled, but the pressure-diffusion term and turbulent-transport term are usually added and modeled as one term,
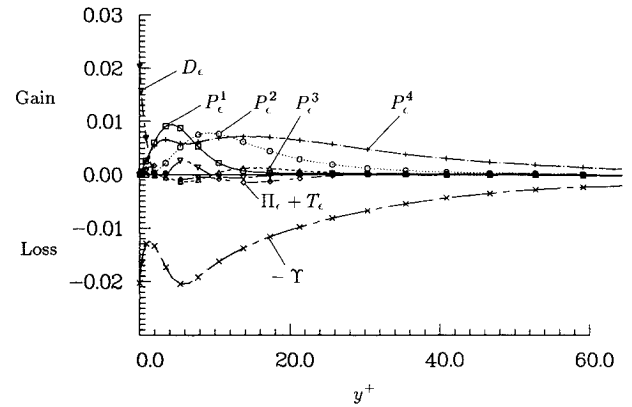
$$T_k + \Pi_k = (\nu_T k_{,j})_{,j} \tag{11}$$

Fig. 2 Terms in the budget of the dissipation rate of the turbulence kinetic energy ϵ in wall coordinates: P_ϵ^1 = mixed production; P_ϵ^2 = production by mean velocity gradient; P_ϵ^3 = gradient production; P_ϵ^4 = turbulent production; T_ϵ = turbulent transport; D_ϵ = viscous diffusion; Υ = dissipation rate; Π_ϵ = pressure transport.
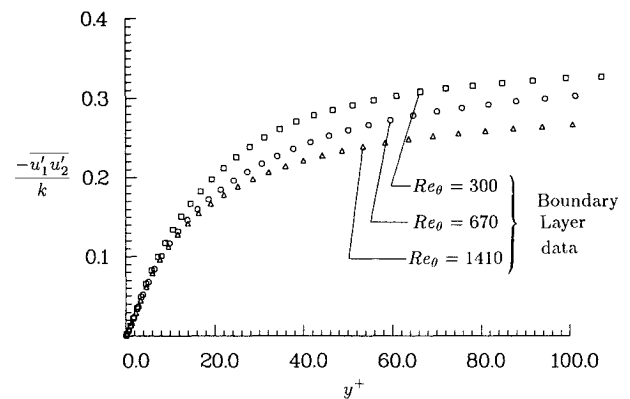

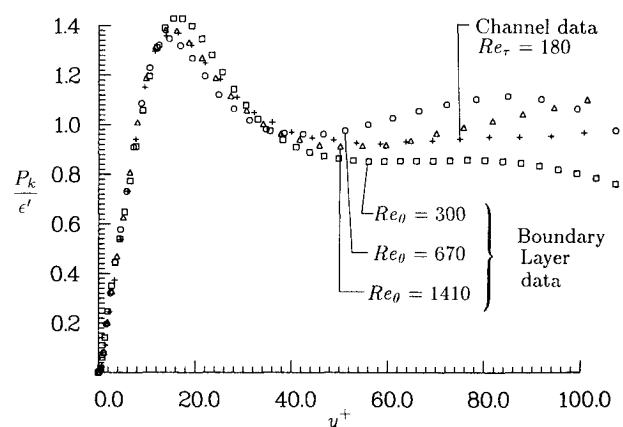Fig. 3 Distribution of $-\overline{u_1'u_2'}/k$ in wall-bounded flows. Boundary-layer data is from Spalart.[10]

Fig. 4 Distribution of P_k/ϵ' in wall bounded flows. Channel data is from Kim et al.[11] and boundary-layer data is from Spalart.[10]

Figure 5 shows the distribution of the turbulence transport term compared to the eddy-viscosity model using $\nu_T = -u_1'u_2'/U_{,2}$ from the data. In the vicinity of the wall, the model has a different slope than the data would indicate; this also can be shown from a Taylor-series expansion in the vicinity of the wall.

The production term is modeled by substituting the model for $-u_i'u_j'$ in the production rate expression,

$$-\overline{u_i'u_j'}\,U_{i,j} = \nu_T 2S_{ij}S_{ij} \tag{12}$$

In this case, the data are matched exactly.

C.　ϵ-Balance

The ϵ-equation is closed by modeling the terms in Eq. (6). To represent the first two production terms, we need an expression for $\overline{u_{i,k}'u_{j,k}'}$ and $\overline{u_{k,i}'u_{k,j}'}$ in terms of k, ϵ and other mean quantities. Note that these two terms have the same trace; they are related for homogeneous flows through the vorticity fluctuation. If we assume that Rotta's approximation is valid for both terms,

$$-(\overline{u_{i,k}'u_{j,k}'} + \overline{u_{k,i}'u_{k,j}'}) = -C_1\frac{\overline{u_i'u_j'}}{k}\epsilon \tag{13}$$

and substitute the Boussinesq approximation for the Reynolds stresses, we obtain:

$$P_\epsilon^1 + P_\epsilon^2 = C_1\frac{\epsilon}{k}\nu_T 2S_{ij}S_{ij} \tag{14}$$

This expression is the same as the commonly used model for the production of ϵ; Fig. 6 shows the model compared to the exact expression, using the constant recommended by CH, $C_1 = 1.35$. The model yields a lower peak than the data would indicate. For near-wall models, the common approach[6] is to introduce a damping function,

$$f_{\epsilon 1}^{1+2} \equiv \frac{P_\epsilon^1 + P_\epsilon^2}{C_1 C_\mu k 2S_{ij}S_{ij}} \tag{15}$$

The term P_ϵ^3 is negligible compared to the other terms in the channel flow, so no explicit expression is used to model it. We expect the turbulent production term P_ϵ^4 to be nonnegligible even in isotropic flows; an appropriate model for this term will be a function of k and ϵ. Dimensional analysis yields $P_\epsilon^4 \propto \epsilon 2/k$. The same arguments are used in modeling the dissipation rate of ϵ as $\propto \epsilon^2/k$. The two terms are then combined and modeled proportional to ϵ^2/k. However, near the wall, $\epsilon^2/k \mapsto \infty$ and the term should be modified for near-wall effects. This is achieved by using the modified dissipation $(\epsilon'' = \epsilon - 2[(k^{1/2}),_2]^2)$ of J&L and setting $P_\epsilon^4 - \Upsilon \propto \epsilon\epsilon''/k$. In this case ϵ''/k and ϵ are bounded as $y^+ \mapsto 0$. Hanjalić and Launder[14] proposed a similar model, and from experimental data of grid turbulence, they inferred that the proportionality factor should be a function of the turbulence Reynolds number ($Re_t = k^2/\epsilon$); therefore, they added a damping function of Re_t. We found (by examining the channel data) that this damping function is an unnecessary complication to the model. The model for the combined terms is given as

$$P_\epsilon^4 - \Upsilon = -C_2\frac{\epsilon\epsilon''}{k} \tag{16}$$

Figure 7 shows the comparison of this model with the data and with the model of Hanjalić and Launder ($C_2 = 1.8$ in their model). The difference between the models is smaller than the difference between the data and the models. The models adequately compare with the data in the $y^+ > 11$ range but underpredict the data close to the wall. We have seen in Fig. 6 that the production rate is underpredicted in the $y^+ < 15$ region, and as we will see from the computational results, the underprediction of both the dissipation rate and the production rate are the major source of discrepancy between the predictions and the data. The common approach in near-wall models is to introduce, in conjunction with a damping function for the production rate of ϵ, a damping function for the dissipation rate,
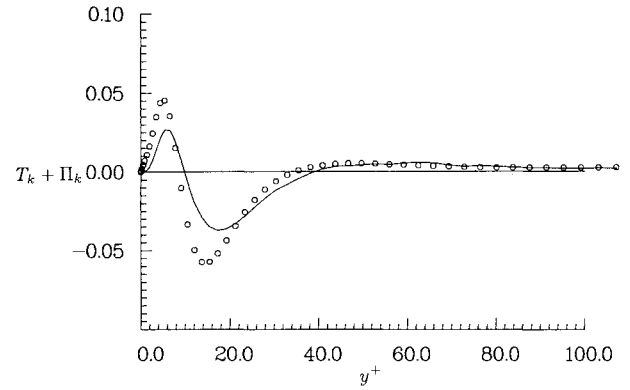

Fig. 5　Transport terms $T_k + \Pi_k$ across the channel: o o o o term computed from the channel data[11]; —— model, Eq. (11).
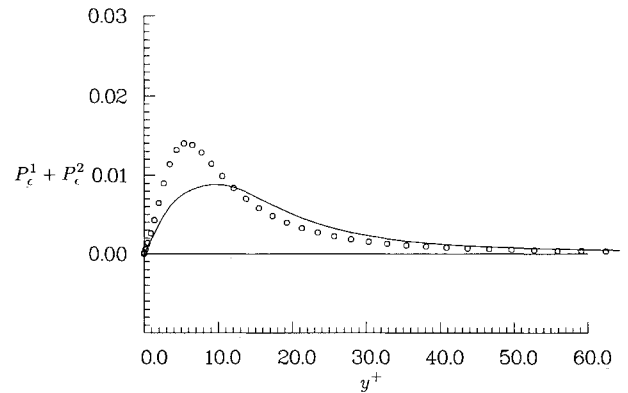

Fig. 6　Production of the dissipation rate of the turbulence kinetic energy $P_\epsilon^1 + P_\epsilon^2$: o o o o term computed from the channel data[11]; —— model, Eq. (14).
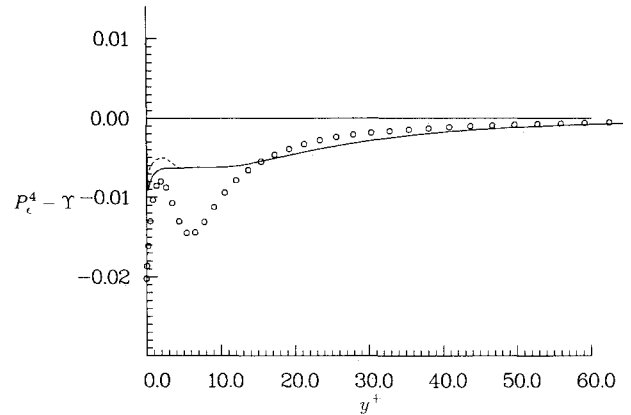

Fig. 7　Net dissipation rate of ϵ, $P_\epsilon^4 - \Upsilon$: o o o o term computed from the channel data[11]; —— model, Eq. (16) ---- model of Hanjalić and Launder.[14]

$$f_{\epsilon 2} \equiv \frac{\Upsilon - P_\epsilon^4}{C_2\epsilon\epsilon''/k} \tag{17}$$

The remaining terms in the balance equation of ϵ are transport rate terms, which are grouped together and modeled using an eddy-viscosity-diffusion model,

$$T_\epsilon + \Pi_\epsilon = \left(\frac{\nu_T}{\sigma}\epsilon_{,j}\right)_{,j} \tag{18}$$

where $\sigma = 1.3$ (J&L and CH). Figure 8 compares the model with the terms representing the left-hand side of Eq. (18) As
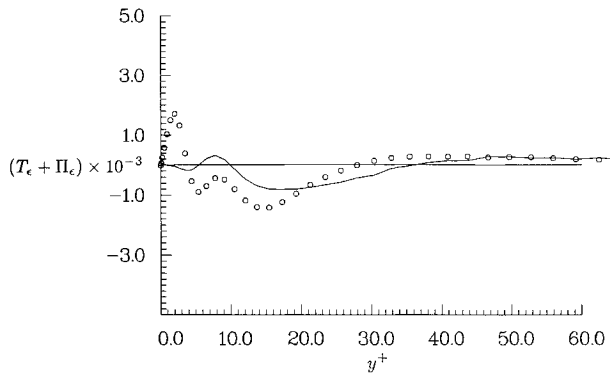
Fig. 8 Turbulent transport rate of ϵ, $T_\epsilon + \Pi_\epsilon$: o o o o term computed from the channel data[11]; —— model, Eq. (18).

in the case for the k-equation, the comparison is not good in the vicinity of the wall; in fact, it can be shown from Taylor series expansion that this model does not have the proper asymptotic behavior as $y^+ \mapsto 0$.

IV. Results Using k-ϵ Models

In our model testing in the previous section, we assumed that we know the distribution of U, k and ϵ; we then tested the model expression. If the models agreed perfectly with the data for all the terms, and if the resulting set of equations admit a unique solution, we can expect the computational results to yield accurate predictions. In general, the agreement is not perfect. For example, the production rate and dissipation rate of ϵ are not well modeled in the $y^+ < 15$ region; therefore we have no assurance that the model will yield accurate predictions. In this section, we will compute the channel flow using the k-ϵ model described in the previous section using the eddy-viscosity-damping function from the data (obtained using uv, k, ϵ, and $U_{,2}$ from the data) and no damping functions in the ϵ-equation.

In our prediction of the channel flow, we begin with the averaged Navier-Stokes equations and simplify them for the case of a fully developed channel. The boundary conditions used are $U = 0$, $k = 0$, $\epsilon = k_{,22}$ at the wall and $U_{,2} = k_{,2} = \epsilon_{,2} = 0$ at the centerline ($y^+ = Re_\tau = 180$). The equations were solved on a nonuniform mesh using the fourth-order Runga-Kutta method to discretize the equations in time and central differencing for space discretization. The same numerical scheme was used to carry out three calculations using the k-ϵ models of J&L, CH, and the model described in Sec. III with the eddy-viscosity damping function from the data.

The mean velocity profiles, as predicted by all models, give acceptable results, but using the exact damping function gives a better overall agreement with the data. Figure 9 shows the k-profiles as predicted by the near-wall models. The model of J&L does not reproduce the sharp peak in the kinetic energy expected near the wall. The model recommended by CH predicts a peak at a slightly shifted location, but the level of k is overpredicted in the $y^+ > 20$ range. When the damping function from the data is used, the peak level is reproduced at approximately the same y^+. The level of k is also better predicted in the rest of the channel. These predictions indicate that in the context of k-ϵ modeling the ad hoc eddy-viscosity damping function plays a key role in the prediction of the k-profile.

Figure 10 shows the profiles of ϵ from the three computations. We find that the models of CH and J&L predict the same total-dissipation-rate ϵ. Near the wall, all models fail to predict ϵ correctly. The models predict a peak in ϵ at $y^+ \approx 10$ whereas the data shows that ϵ peaks at the wall. However, it is interesting to note that the peak predicted by the models is
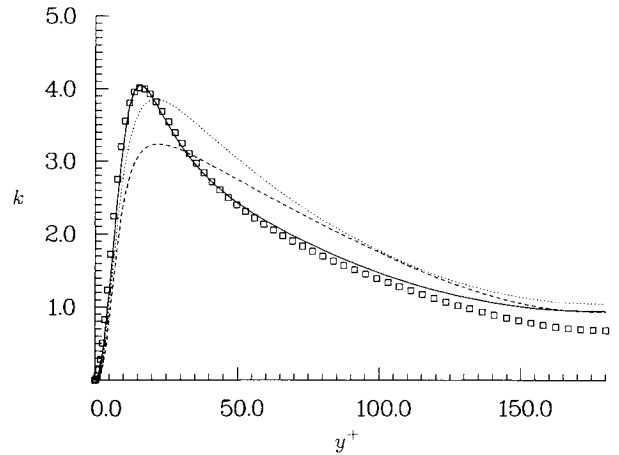

Fig. 9 Turbulence kinetic energy distribution for $Re_\tau = 180$: ▫ ▫ ▫ ▫ term computed from the channel data; —— model using f_μ from the data, ······· Chien's model,[4] ---- Jones and Launder's model.[3]
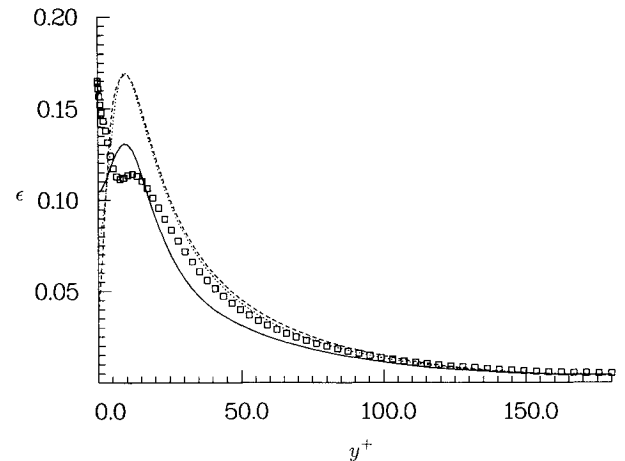
Fig. 10 Dissipation rate (ϵ) distribution for $Re_\tau = 180$: ▫ ▫ ▫ ▫ term computed from the channel data; —— model using f_μ from the data, ·······Chien's model,[4] ---- Jones and Launder's model.[3]
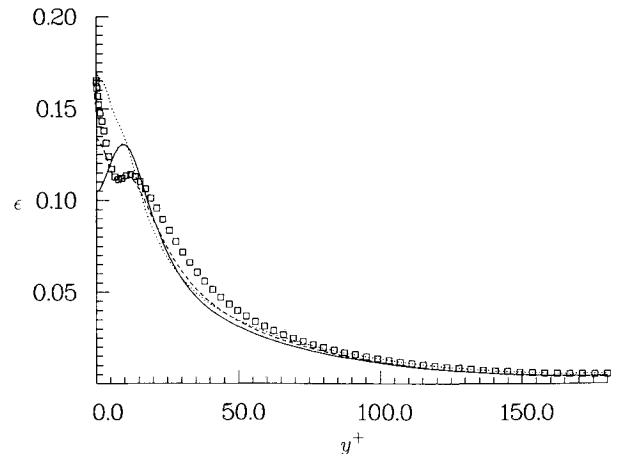
Fig. 11 Dissipation rate (ϵ) distribution for $Re_\tau = 180$: ▫ ▫ ▫ ▫ term computed form the channel data[11]; —— model using f_μ from the data with $f_{c1}^{1+2} = f_{c2} = 1$; ······· model with f_μ, f_{c1}^{1+2}, and f_{c2} from the data; ---- model with f_μ, f_{c1}^{1+2+3}, and f_{c2} from the data.

at the same location where the data show a local peak. This comparison indicates that the damping functions in the ϵ-equation will play an important role in our prediction of the correct ϵ-profile.

V.   Conclusions and Discussion

In this paper, we used turbulence data from a full simulation to compute the terms in the budgets of the turbulent kinetic energy and of the dissipation rate.

These budgets were used to test closure models for the k and ϵ equations. We find that the transport models need to be improved near the wall. The closures to the ϵ-equation are good away from the wall but poor in the vicinity of the wall. We have split the eddy-viscosity damping function into two functions; one representing the ratio of the production over a modified ϵ, which we find to be a function of y^+ only, the other representing the parameter $-\overline{u_1'u_2'}/k$, which we find to be a function of y^+ and the Reynolds number.

Computations with the k-ϵ model indicate that the ad hoc eddy-viscosity damping function plays a key role in the prediction of the k-profile. This requirement should be considered as a limitation of k-ϵ modeling. To improve the prediction of the ϵ-profile, wall corrections in the ϵ-equation need to be introduced. In fact, using the damping functions $f_{\epsilon1}^{1+2}$ and $f_{\epsilon2}$ from the data gives a better prediction for the ϵ-profile (see Fig. 11). However, the improvement is not complete. By neglecting P_ϵ^3, we effectively have lumped the term with the model of $P_\epsilon^1 + P_\epsilon^2$ and the damping function for the production term should be

$$f_{\epsilon1}^{1+2+3} \equiv \frac{P_\epsilon^1 + P_\epsilon^2 + P_\epsilon^3}{C_1 C_\mu k \, 2 S_{ij} S_{ij}} \qquad (19)$$

Use of the above damping function yields a better ϵ-profile (see Fig. 11). The remainder of the disagreement must be due to the model of the transport terms.

Acknowledgments

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